SOLUTION OF A GENERALIZED PROBLEM OF HEAT OR MASS TRANSFER IN A BED

G. A. Aksel'rud

Inzhenerno-Fizicheskii Zhurnal, Vol. 11, No. 1, pp. 93–98, 1966

UDC 536.24

The kinetics of unrelated heat and mass transfer processes are examined for the case of motion of a liquid or a gas through a bed. The solution obtained applies to particles of various shapes (infinite plate, infinite cylinder, sphere).

Heat or mass transfer during motion of liquid or gas through a bed has been examined by many investigators in connection with processes found in heating of a charge or batch, surface burning, drying, diffusion extraction from a porous bed, absorption, and ion exchange. The early solutions [1, 2] were based on the following assumptions.

1) Thermal or diffusion resistance within the particles may be neglected.

2) Contact thermal conduction or diffusion may also be neglected.

3) Transfer of heat or mass in the direction of the bed length occurs only because of motion of the medium through the bed. Reverse heat conduction or diffusion, due to temperature or concentration gradients plays no appreciable role.

4) The particles comprising the bed are spherical in shape.

Of all these assumptions, condition 1 is evidently not valid in many cases. In this connection, many authors have solved the problem under examination with allowance for both internal and external heat or mass transfer [3–7]. As far as condition 4 is concerned, i.e., spherical particles, it also is not always fulfilled. Thus, in extraction processes from vegetable raw material, the shape of the body mostly conforms with the concept of "infinite plate." Sugar-beet shavings, from which sugar is extracted, have a length tens of times their width or thickness, the latter two dimensions being similar to each other. This shape may be likened to an infinite cylinder.

In the present paper we have obtained the equations of the kinetics of unrelated heat and mass transfer processes in a bed, with allowance for both external and internal heat or mass transfer, valid for each of the following three shapes: infinite plate, infinite cylinder, and sphere. The starting point is the following system, consisting of a differential equation and boundary conditions:

$$\frac{\partial U(x, t')}{\partial t'} = D\left[\frac{\partial^2 U(x, t')}{\partial x^2} + \frac{\Gamma}{x}\frac{\partial U(x, t')}{\partial x}\right]; \quad (1)$$

$$U(x, 0) = U_0; \quad (2)$$

$$\frac{\partial U(0, t')}{\partial x} = 0; \quad (3)$$

$$-D_1\frac{\partial U(R, t')}{\partial x} = k[U(R, t') - U_1(t', z)]; \quad (4)$$

$$v\frac{\partial U_1(t', z)}{\partial z} + \sigma\frac{\partial \bar{U}(t', z)}{\partial t'} = 0; \quad (5)$$

$$U_1(t', 0) = U_{in}. \quad (6)$$

In this system, Eqs. (1)–(4) constitute the condition of the usual problem of unsteady heat conduction or diffusion with boundary conditions of the third kind [8]. The temperature or mass content of the external medium varies with time. This variation is described by an equation of heat or mass balance (5) and the corresponding boundary condition (6). Since the function under study in the external medium varies along the length of the bed z, even within the range of each particle, the field of this quantity will depend on the position of the particle, relative to the section at which the moving medium enters.

We introduce the following group of dimensionless quantities:

$$\varphi = x/R, \quad \tau = Dt'/R^2, \quad \mathrm{Bi} = kR/D_1, \quad \omega = \sigma Dz/vR^2,$$

whereupon the system being examined takes the form

$$\frac{\partial U(\varphi, \tau)}{\partial \tau} = \frac{\partial^2 U(\varphi, \tau)}{\partial \varphi^2} + \frac{\Gamma}{\varphi}\frac{\partial U(\varphi, \tau)}{\partial \varphi};$$

$$U(\varphi, 0) = U_0;$$

$$\frac{\partial U(0, \tau)}{\partial \varphi} = 0;$$

$$-\frac{\partial U(1, \tau)}{\partial \varphi} = \mathrm{Bi}[U(1, \tau) - U_1(\tau, \omega)];$$

$$\frac{\partial U_1(\tau, \omega)}{\partial \omega} + \frac{\partial \bar{U}(\tau, \omega)}{\partial \tau} = 0;$$

$$U_1(\tau, 0) = U_{in}.$$

Solution is accomplished by a Laplace transform method, for which we introduce transforms of the desired functions:

$$F(\varphi, p) = \int_0^\infty \exp(-p\tau) U(\varphi, \tau)\, d\tau,$$

$$T(p, \omega) = \int_0^\infty \exp(-p\tau) U_1(\tau, \omega)\, d\tau.$$

$$pF(\varphi, p) - U_0 = \frac{d^2F(\varphi, p)}{d\varphi^2} + \frac{\Gamma}{\varphi}\frac{dF(\varphi, p)}{d\varphi}; \quad (7)$$

$$\frac{dF(0, p)}{d\varphi} = 0; \quad (8)$$

$$-\frac{dF(1, p)}{d\varphi} = \mathrm{Bi}[F(1, p) - T(p, \omega)]; \quad (9)$$

$$\frac{dT(p, \omega)}{d\omega} + p\bar{F}(p, \omega) - U_0 = 0; \tag{10}$$

$$T(p, 0) = U_{in}/p. \tag{11}$$

A solution of (7) satisfying condition (8) is the function

$$F(\varphi, p, \omega) = U_0/p + G(\omega)(\sqrt{p}\varphi)^{-\nu} I_\nu(\sqrt{p}\varphi)$$

and the functions deriving from it

$$F(1, p, \omega) = U_0/p + G(\omega) p^{-\nu/2} I_\nu(\sqrt{p}),$$

$$\bar{F}(p, \omega) = \frac{1}{V}\int_{(v)} F dV = U_0/p + G(\omega)\cdot 2p^{-(\nu+1)/2}(1+\nu) I_{\nu+1}(\sqrt{p}),$$

where

$$\frac{dV}{V} = (2\nu+2)\varphi^{2\nu+1} d\varphi;$$

$\nu = (\Gamma - 1)/2$; $\nu = -1/2$ for an infinite plate;
$\nu = 0$ for an infinite cylinder;
$\nu = +1/2$ for a sphere.

After substituting the expressions obtained into (9) and (10), and with the help of boundary condition (11), we may determine the constant $G(\omega)$ and the quantity $T(p, \omega)$. This latter quantity is of very great interest, since it is the transform of the function describing the state of the medium moving through the bed

$$\frac{U_0/p - T(p, \omega)}{U_0 - U_{in}} = \frac{1}{p}\exp[-\omega M(\mathrm{Bi}, \sqrt{p})]; \tag{12}$$

$$M(\mathrm{Bi}, \sqrt{p}) = \frac{2(1+\nu)}{1/\mathrm{Bi} + I_\nu(\sqrt{p})/\sqrt{p} I_{\nu+1}(\sqrt{p})}. \tag{13}$$

For transformation to the original we use the Rosen [5] method of transformation of the contour integral

$$\frac{U_0 - U_1}{U_0 - U_{in}} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{1}{p}\exp[p\tau - M(\mathrm{Bi}, \sqrt{p})\omega]\, dp. \tag{14}$$

Since all the singular points of the integrand lie to the left of the origin, except for the one which coincides with the origin, the path of integration is chosen along the imaginary axis, by passing the point p = 0:

$$\frac{U_0 - U_1}{U_0 - U_{in}} = \frac{1}{2\pi i}\lim_{\varepsilon \to 0}\left[\int_{-i\infty}^{-i\varepsilon} + \int_c + \int_{i\varepsilon}^{+i\infty}\right].$$

The integral along the contour bypassing the point p = = 0 is easily determined:

$$\frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_c \frac{1}{p}\exp[p\tau - \omega M(\mathrm{Bi}, \sqrt{p})]\, dp = \frac{1}{2\pi i}\int_c \frac{dp}{p} = \frac{1}{2}.$$

To transform the first and third integrals, the change of variables p = −iy and p = +iy is used. Then

$$\frac{U_0 - U_1}{U_0 - U_{in}} = \frac{1}{2} + \frac{1}{2\pi}\int_0^\infty [f(iy) + f(-iy)]\, dy;$$

$$f(p) = \frac{1}{p}\exp[p\tau - \omega M(\mathrm{Bi}, \sqrt{p})].$$

In what follows we use the following properties of the integrand:

$$f(\bar{p}) = \bar{f}(p), \quad f + \bar{f} = 2R(f).$$

In these relations p and $\bar{p}$, f and $\bar{f}$ are complex interconnected quantities;

$$\frac{U_0 - U_1}{U_0 - U_{in}} = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty R[f(iy)]\, dy.$$

The final result takes the form

$$\frac{U_0 - U_1}{U_0 - U_{in}} = \frac{1}{2} + \frac{2}{\pi}\int_0^\infty \exp(A_1\omega)\sin(B_1\omega + \lambda^2\tau)\frac{d\lambda}{\lambda}, \tag{15}$$

where

$$y = \lambda^2;$$

$$A_1 = \left[-\left(\frac{1}{\mathrm{Bi}} + \frac{A - B}{\sqrt{2}\lambda}\right)2(1+\nu)\right] \Big/ \left[\left(\frac{1}{\mathrm{Bi}} + \frac{A-B}{\sqrt{2}\lambda}\right)^2 + \frac{(A+B)^2}{2\lambda^2}\right];$$

$$B_1 = \left[\frac{\sqrt{2}}{\lambda}(A+B)(1+\nu)\right] \Big/ \left[\left(\frac{1}{\mathrm{Bi}} + \frac{A-B}{\sqrt{2}\lambda}\right)^2 + \frac{(A+B)^2}{2\lambda^2}\right];$$

$$A = \frac{ber_\nu\lambda\, ber_{\nu+1}\lambda + bei_\nu\lambda\, bei_{\nu+1}\lambda}{ber^2_{\nu+1}\lambda + bei^2_{\nu+1}\lambda};$$

$$B = \frac{ber_{\nu+1}\lambda\, bei_\nu\lambda - ber_\nu\lambda\, bei_{\nu+1}\lambda}{ber^2_{\nu+1}\lambda + bei^2_{\nu+1}\lambda};$$

and ber λ, bei λ are Thomson's functions.

To calculate the quantities A and B it is convenient to use the relations obtained by Whitehead [9]:

$$ber_r\lambda\, ber_\rho\lambda + bei_r\lambda\, bei_\rho\lambda = \sum_{n=0}^\infty (-1)^n\cos\left[\frac{3}{2}(r-\rho) - n\right] \times \frac{\pi}{2}\cdot B_{r,\rho,n}\left(\frac{\lambda}{2}\right)^{r+\rho+2n},$$

$$bei_r\lambda\, ber_\rho\lambda - ber_r\lambda\, bei_\rho\lambda = \sum_{n=0}^\infty (-1)^n\sin\left[\frac{3}{2}(r-\rho) - n\right] \times \frac{\pi}{2}\cdot B_{r,\rho,n}\left(\frac{\lambda}{2}\right)^{r+\rho+2n},$$

$$ber^2_\rho\lambda + bei^2_\rho\lambda = \sum_{n=0}^\infty (-1)^n\cos\frac{\pi n}{2} B_{\rho,\rho,n}\left(\frac{\lambda}{2}\right)^{2\rho+2n},$$

$$B_{r,\rho,n} = \sum_{m=0}^n \frac{(-1)^m}{m!(n-m)!\,\Gamma(r+n-m+1)\Gamma(\rho+m+1)},$$

in which we must put $r = \nu$ and $\rho = \nu + 1$.

The solution represented by the integral (15) is exact, though it is inconvenient for direct calculations. We will indicate two approximate results.

For $\omega \ll 1$, $\tau > 0$ it follows from (12) and (13) that

$$\frac{U_0/p - T(p, \omega)}{U_0 - U_{in}} = \frac{1}{p}\left[1 - \frac{2(1+\nu)}{1/\mathrm{Bi} + I_\nu(\sqrt{p})/\sqrt{p}\, I_{\nu+1}(\sqrt{p})}\right],$$

and going over to the transform leads to the result

$$\frac{U_0 - U_1}{U_0 - U_{in}} = 1 - \omega \sum_{n=1}^{\infty} \frac{4(\nu+1)\mathrm{Bi}^2}{\mathrm{Bi}(\mathrm{Bi} - 2\nu) + \mu_n^2} \exp(-\mu_n^2 \tau), \quad (16)$$

where μ_n are roots of the characteristic equation

$$J_\nu(\mu)/J_{\nu+1}(\mu) = \mu/\mathrm{Bi}.$$

For $\omega \gg 1$, analysis of A_1 and B_1 shows that these may be represented by the power series

$$A_1 = -\frac{1}{2(\nu+1)}\left[\frac{1}{\mathrm{Bi}} + \frac{1}{2(\nu+2)}\right]\lambda^4 + \ldots,$$
$$B_1 = -\lambda^2 + \ldots. \quad (17)$$

For large values of ω, integral (15) leads to an asymptotic estimate. The main contribution comes from those values of the integrand which are determined with small λ. Substituting (17) and (15) and integrating gives the following result:

$$\frac{U_0 - U_1}{U_0 - U_{in}} = \frac{1}{2}\left[1 + \mathrm{erf}\,\frac{\tau - \omega}{\sqrt{\omega/(\nu+1)[2/\mathrm{Bi} + 1/(\nu+2)]}}\right]. \quad (18)$$

Thus, the exact solution (15) for the problem of transfer of heat or mass in a bed is valid for each of three shapes—infinite plate, infinite cylinder, and sphere. Approximate solutions (16) and (18) have been established for the same problem.

NOTATION:

Bi) Biot number; D) thermal diffusivity or concentration conductivity; D_1) thermal or mass conductivity; F) transform of the function U; k) heat or mass transfer coefficient; p) Laplace transform parameter; R) particle dimension (half-width of plate, radius of cylinder, radius of sphere); t) time reckoned from moment of arrival of moving medium at given point in bed; U) temperature or concentration; U_0) the same at time zero; U_1) temperature or concentration of the medium moving through the bed; U_{in}) the same at bed inlet section; V) particle volume; v) rate of flow of fluid through bed; x) variable coordinate within the range of a particle; z) distance of any point of bed from inlet section; T) transform of the function U_1; y) imaginary part of the parameter p; Γ) shape constant, equal to zero for an infinite plate, 1 for a cylinder, and 2 for a sphere; ε) radius of semicircle bypassing origin of coordinates (the point p = 0); λ) variable of integration; μ_n) roots of characteristic equation; $\nu = (\Gamma - 1)/2$) shape constant; σ) ratio of water equivalent of unit volume of bed to specific heat of liquid, for heat transfer, and volume of pores in unit volume of the bed, for mass transfer; τ) dimensionless time; φ) dimensionless variable coordinate; ω) dimensionless bed length.

REFERENCES

1. A. Anzelius and Z. Angew, Math. und. Mech., **6**, no. 4, 291, 1926.

2. T. E. Schumann, J. of the Franklin Inst., **208**, 405, 1929.

3. G. P. Ivantsov and B. Ya. Lyubov, DAN SSSR, **86**, no. 2, 1952.

4. E. Wicke, Z. Kolloid, **93**, 129, 1940.

5. J. B. Rosen, Journ. Chem. Phys., **20**, no. 3, 387, 1952.

6. J. B. Rosen, Ind. Eng. Chem., **46**, no. 8, 1954.

7. G. A. Aksel'rud, Theory of Diffusion Extraction of Materials from Porous Bodies [in Russian], L'vov, 1959.

8. A. V. Luikov and Yu. A. Mikhailov, Theory of Heat and Mass Transfer [in Russian], GEI, 1963.

9. C. S. Whitehead, Quart. Journ. Math., **42**, 316, 1911.

6 September 1965

L'vov Polytechnic Institute